PHIL 4310: Advanced Logic

Spring 2026 — Homework 3

Reading: Finish reading Chapter 2 of *Logic for Philosophy*.

Part I The Soundness Theorem

1. Show that the $\to$E rule is validity preserving.

2. Show that the $\to$I rule is validity preserving.

3. Imagine that we added rules to Sider's sequent system. Would the Soundness Theorem continue to hold if we added the following rules? (*These are two separate questions—the rules are not both being added simultaneously.*) Explain your answers.

$$\frac{\Delta \Rightarrow \varphi \to \psi \qquad \Sigma \Rightarrow \chi \to \psi}{\Delta, \Sigma \;\Rightarrow\; (\varphi \vee \chi) \to \psi} \ (\vee{\to}\text{Intro})$$

$$\frac{\Delta \Rightarrow (\varphi \leftrightarrow \psi) \to \chi \qquad \Sigma \Rightarrow \varphi \to \chi}{\Delta, \Sigma \;\Rightarrow\; \psi \to \chi} \ (\leftrightarrow\,{\to}\text{Elim})$$

4. Unsound rules are very bad. Imagine that we thought that $\vee$ worked like '*exactly one*' instead of '*at least one*'. We might then think that the following rule would be acceptable:

$$\frac{\Delta \Rightarrow \varphi \qquad \Sigma \Rightarrow \varphi \vee \psi}{\Delta, \Sigma \;\Rightarrow\; {\sim}\psi} \ (\text{Ex-}{\vee}\text{Elim})$$

Show that if we add this rule to Sider's proof system, we can derive $P \Rightarrow {\sim}P$.

Extra credit: In fact, adding this rule allows us to prove *any* possible sequent. Prove that this is true.

Extra, extra credit: Moreover, adding *any* unsound rule with metavariables in the style of the other Sider rules makes every possible sequent derivable. Prove this.

5. More on Soundness and Completeness. Prove that for any formula φ, *exactly one* of the following three conditions holds:

(a) $\{\,\} \Rightarrow \varphi$ is a valid sequent.

(b) $\{\,\} \Rightarrow \sim\varphi$ is a valid sequent.

(c) There is a PL interpretation where $V(\varphi) = 1$ *and* a PL interpretation where $V(\sim\varphi) = 1$.

For this problem you may assume without argument that both the Soundness Theorem and the Completeness Theorem hold for PL. If you use either theorem, indicate clearly *which* theorem you are using and *exactly how* you are using it in your proof.

Part II Consequences of a Valid Sequent

Assume that $\{P_1, P_2, P_3\} \Rightarrow Conc$ is a valid sequent, i.e. that *Conc* is derivable from $\{P_1, P_2, P_3\}$.

Which of the following **MUST** be true? (The correct answer may include any number of these statements.)

1. *Conc* is a logical consequence of $\{P_1, P_2, P_3\}$.

2. $\sim Conc$ is *not* a logical consequence of $\{P_1, P_2, P_3\}$.

3. $\{P_1, P_2, P_3\}$ is a consistent set.

4. $\{P_1, P_2, P_3\}$ is an inconsistent set.

5. $\{P_1, P_2, P_3, Conc\}$ is an inconsistent set.

6. $\{P_1, P_2, P_3, \sim Conc\}$ is an inconsistent set.

7. $\{P_2, P_3, \sim Conc\}$ is an inconsistent set.

8. $\{P_2, P_3, \sim Conc\}$ is a consistent set.

9. $\{\sim P_1, P_2, P_3, Conc\}$ is an inconsistent set.

10. $\{\sim P_1, \sim P_2, \sim P_3, Conc\}$ is a consistent set.

11. $\sim P_1$ is a logical consequence of $\{P_2, P_3, Conc\}$.

12. $\sim P_1$ is a logical consequence of $\{P_2, P_3, \sim Conc\}$.

13. $\sim P_3$ is derivable in from $\{P_1, P_2, \sim Conc\}$.

14. P_3 is derivable from $\{P_1, P_2, Conc\}$.

15. $P_1 \to Conc$ is derivable from $\{P_2, P_3\}$.

16. $P_1 \leftrightarrow Conc$ is derivable from $\{P_2, P_3\}$.

17. $\sim Conc \to \sim P_3$ is derivable from $\{P_1, P_2\}$.

18. $(P_1 \wedge P_2 \wedge P_3) \to Conc$ is derivable from $\{\}$.

19. $(\sim P_1 \wedge \sim P_2 \wedge \sim P_3) \to \sim Conc$ is *not* derivable from $\{\}$.

20. $P_1 \to (P_2 \to (P_3 \to Conc))$ is a logical truth.

21. $\sim Conc \to (\sim P_1 \wedge \sim P_2 \wedge \sim P_3)$ is a logical truth.

Part III What Must Be False?

Which of the 21 sentences above **MUST** be false? (*Hint:* The answer is not just everything that wasn't correct in Part II.)

Part IV Proofs by Induction

Use structural induction to prove the following claims.

1. Every well-formed formula has the same number of left parentheses as right parentheses.

2. Theorem (Substitution). Let φ and ψ be logically equivalent formulas. We'll write this as $\varphi \models\!\!\dashv \psi$. Now let χ be any formula, and let χ' be the result of replacing zero or more occurrences of φ in χ by ψ. Show that $\chi \models\!\!\dashv \chi'$.

3. Duality. Let α be a wff whose only connective symbols are $\wedge$, $\vee$, and $\sim$. Let α^* be the result of interchanging every $\wedge$ with $\vee$ (and vice versa) and replacing each sentence symbol P by $\sim P$. Show that $\alpha^* \models\!\!\dashv \sim\alpha$.

Remark: It follows that if $\alpha \equiv \beta$ then $\alpha^* \models\!\!\dashv \beta^*$.

4. Let φ and ψ be wffs whose *only* connectives are $\sim$ and $\leftrightarrow$, and which have the same sentence letters (with the same number of occurrences of each) and the same number of $\sim$ symbols and the same number of $\leftrightarrow$ symbols. Show that $\varphi \models\!\!\dashv \psi$.

Examples:

- $\varphi = \sim(A \leftrightarrow B)$ and $\psi = (A \leftrightarrow \sim B)$
- $\varphi = \sim(\sim A \leftrightarrow B) \leftrightarrow (\sim C \leftrightarrow A)$ and $\psi = (\sim\sim\sim A \leftrightarrow (A \leftrightarrow (B \leftrightarrow C)))$

Part V Knights, Knaves, and Normals

On a certain island there are **knights** who always tell the truth, **knaves** who always lie, and **normals** who sometimes tell the truth and sometimes lie. Knights have the highest rank, normals the middle rank, and knaves the lowest rank.

A says: *"I am of a lower rank than B."*

B says: *"That's not true!"*

Determine what rank each of A and B is, and whether what they say is true.

Encoding into propositional logic.

Now lets show how we can encode all of the information in this problem into propositional logic. Lets use the following atomic sentences:

$$\begin{array}{llll} A_1 & = \text{Knight}(A) & B_1 & = \text{Knight}(B) \\ A_2 & = \text{Knave}(A) & B_2 & = \text{Knave}(B) \\ A_3 & = \text{Normal}(A) & B_3 & = \text{Normal}(B) \end{array}$$

Now using these atomic sentences, write each of the following as a formula of propositional logic.

1. A is exactly one of a knight, knave, or normal.

2. B is exactly one of a knight, knave, or normal.

3. If A is a knight, then A is of a lower rank than B.

4. If A is a knave, then it is *false* that A is of a lower rank than B.

5. If B is a knight, then it is *false* that A is of a lower rank than B.

6. If B is a knave, then A is of a lower rank than B.

Note: "A is of a lower rank than B" must itself be expressed in terms of the atomic sentences above. Think carefully about what combinations of A_i and B_j make the rank claim true.